Time-Frequency Analysis of CAPM: Application to the CAC 40

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The market line estimation implicitly assumes that its parameters are constant over time supposing whatever the investment horizon, the investors have a similar behaviour. In this paper, we discuss this hypothesis using the technique of wavelets. First, we verify the expected result concerning the statistical weaknesses of market line and the high volatility of its parameters. Second, we use the wavelets to estimate the frequency betas. We show that the classic beta (estimated with OLS) considers a short-run beta. We propose a methodology based on time-frequency analysis that leads to an overview of equities characteristics useful to portfolio managers.

Key Words: market line, wavelets, MODWT, frequency betas
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Introduction

Modern portfolio theory is initiated by Markowitz (1952) and it is extended with the Capital Asset Pricing Model (CAPM) by Sharpe (1964), Lintner (1965), and Mossin (1966). The main idea is the systematic risk of an asset related to its link with the market (the system). The CAPM implies that the market is the only risk factor explaining stock prices. The sensibility of the security to the market is measured by the beta (systematic risk). The model was tested in many econometrics studies as Black, Jensen, and Scholes (1972), Fama and MacBeth (1973). These studies highlight the instability of CAPM parameters (including the beta) according to the chosen period or the data frequency. But the validity or the rejection of the CAPM can't be totally proved. The CAPM's validity (or rejection) is always discussed because the beta is widely used by investors as a risk indicator.

The hypothesis of constant beta is one of the biggest CAPM criticisms because it supposes agents have a similar behaviour. They select their assets homogeneously but nothing certifies a similar investment horizon. Some agents are high frequency traders (as banks) using automatic algorithms, whereas mutual funds have a longer investment. So, if we suppose that they invest in the same assets (but with different horizons), in this case they don't face necessarily the same risk level.

According to the frequency, the risk could be different and linked with a time horizon. The investor has a different behaviour so the systematic risk level has differentiated values and dynamics across frequency. This topic is the main subject of Gençay, Selçuk, and Whitcher (2005). They prove the existence of systematic risk at frequency level by using wavelets but they don't study Beta temporal evolution. They also indicate the pertinence of CAPM's relationships at medium and long run.

For testing the Heterogeneous Behaviour hypothesis, we use the frequency decomposition of stock prices enhanced by a temporal location. This is the Maximal Overlap Discret Wavelets Transform or MODWT (Mallat 1989; 2001; 2009; Meyer 1990). We are able to study the behaviour of the chronic by frequency levels while having a temporal location by frequency bands. This process is perfectly appropriate to our problem.

In the case, we use the CAPM of the 30 permanent stocks quoted on the French market CAC 40 for the daily period from January 2005 to December 2015. This period includes calm and turbulent market's periods as the 'subprime' crisis. In a first part, we present the CAPM and we estimate the market's line parameters in order to verify their instability over time. In a second part, we use the wavelets to construct a time-frequency CAPM considering the heterogeneous behaviour of agents. We show that for a portfolio manager, the investment choices are more differentiated by using frequency betas.

Theoretical and Empirical CAPM

CAPM's Literature Overview

The Capital Asset Pricing Model of Sharpe (1964) is based on the Securities Market Line equation (or SML)

$$R_{i,t} = rf + \beta_i(R_{m,t} - rf). \tag{1}$$

The CAPM explains the return of an asset i, $R_{i,t}$, by the market returns $R_{m,t)}$. In portfolio theory, the agents invest in equities (risky assets) and in a risk-free asset with return noted rf. Consequently, the equities have to offer bigger returns than risk-free asset. This is the risk premium noted $r_{i,t}$. From equation (1), we define the Market Premium $r_{m,t}$ as the difference of Market return and risk-free rate. We can write equation (1) with risk premia such as:

$$r_{i,t} = \alpha_i + \beta_i r_{m,t} + \varepsilon_{i,t}. \tag{2}$$

The parameters of equation (2) are estimated by OLS, $\varepsilon_{i,t}$ are residuals of the regression supposed to be a white noise process i.i.d $(0, \sigma_\varepsilon)$. The slope of this line represents the Systematic risk. By construction it is constant over time. Theoretically, α_i is null, it represents the shift between the forecasted value of the expected return and empirical excepted return. This parameter is useful to test the validity of the model.

Concerning the CAPM's literature, many studies focuses on it validity. The first studies, realized in the late 1960s, tend to confirm the model. But the works of Sharpe and Cooper (1972) focus on several statistical biases related to estimation methods. These results are confirmed by others authors such as Black, Jensen, and Scholes (1972), Fama and MacBeth (1973) in USA's stocks markets. However, these authors highlight the instability of parameters (mainly the beta) according to the chosen period and frequency's data. In the European stock markets, these conclusions are similar (Modigliani, Pogue, and Solnik 1972).

In 1992, the biggest criticism of the model is formulated by Fama and French. For these authors the 'beta is dead' because many stylized facts are not considered by the CAPM. Multifactor model was developed such as Arbitrage Pricing Theory or Fama-French Model. In the opposite side, Chan and Lakonishock (1993) consider the Beta is not totally dead because there are not pertinent and sufficient arguments 'to kill' it. So, the CAPM validity or rejection is not totally established and the Beta is widely used.

Many papers focus on the Beta instability phenomenon because it is the major parameter of the CAPM. The studies of Fabozzi and Francis (1978), Bos and Newbold (1984) confirm the Beta instability over time. The beta volatility ascends with the length of the period, because in the returns's structures the OLS method doesn't consider breaks. Since these works, Groenwold and Fraser (1997) revisit the model with different methods to correct the beta. One of these methods estimates the Beta for 5 years period introducing the principle of Rolling Beta Estimate with varying window size. Hawawini (1983) studies the impact of data frequency on the Beta value and gives an adjusted measure to correct the beta. This one is different in the case of daily, weekly or monthly data and it depends on the bias based on the asset's market value.

The beta is related to the horizon and the frequency of the data which is inadequate with the hypothesis of homogeneous behaviour of agents. In practice, agents have different investment horizons so, the beta's estimations must consider these facts. Following this hypothesis, Gençay, Selçuk, and Whitcher (2005) analyse the beta by a wavelets approach with daily data. They conclude that the CAPM is more relevant in medium and long run.

In this paper, we follow the work of Gençay, Selçuk, and Whitcher (2005) in order to appreciate the impact of behavioural hypothesis on the systematic risk in French Stock-Market. We use the log-excess returns of 30 listed companies and the CAC40 index (as the proxy of Market) for the daily period from January 2005 to December 2015. The rate of 'OAT 10 years' is the risk-free rate. In order to catch different investment horizon with the wavelets frequency bands, we use daily data from Yahoo Finance database and Banque de France (French Central Bank) website.

CLASSIC BETA ESTIMATE (PERIOD 2005–2015)

The betas are estimated over the period, then we repeat this procedure over sub-periods. And finally, we study the volatility and instability of betas using rolling regression. In the paper, the Beta estimated by OLS is called the 'Classic Beta' opposite to the 'Frequency Betas' estimated by Wavelets.

Before estimation, we test the stationary character of the variables with the Philips-Perron test (and KPSS). Residuals of the OLS estimation are autocorrelated, heteroscedastic and non-Gaussian. The minimum variance property is not respected. The regression is repeated using Quasi-Generalized Least Squares (GLS) with the Newey-West matrix (1987) more robust to unknown forms of autocorrelation and heteroscedasticity.

Table 1 summarizes the results of GLS estimations and provides the R-squared and usual tests on parameters and residuals. We accept the significance of Betas and R^2, but the residuals are autocorrelated and heteroscedastic with a strong non-normality for the majority of equities. Despite these weaknesses, using betas, it is possible to differentiate and characterize easily stocks. For example, financial stocks have a high beta compared to the communication sector stocks like Vivendi and Publicis. It is therefore possible, to classify equities by the beta, its characteristics and its temporal evolution. If the Beta is greater than 1, the equity am-

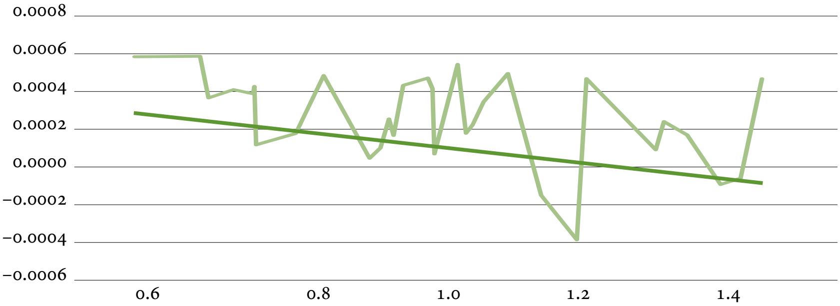

FIGURE 1 Return-Beta Relation

plifies the markets movements. If it is lesser than 1 1, the stock attenuates the market fluctuations. The value of the beta is important for a portfolio manager to appreciate the systematic risk. According to the CAPM theory, a stock with a high beta (high systematic risk) must have consequently a higher expected return than a stock with low beta. Figure 1 represents in the ordinate the mean return of the asset versus the Betas in the abscissa.

ANALYSIS OF CLASSIC BETA INSTABILITY

By analysing the result of figure 1, we note that the equities with high beta have a lower risk premium compared to low beta stocks. These results imply a decreasing SML contrary to the CAPM hypothesis. The time period includes many shocks such as the financial crisis (2008), the debt crisis (2011–2012) and may cause this result. So, our original sample is divided into three, in order to consider different periods: the first one stretches from January 2005 to December 2007 corresponding to a 'calm' period where the market is expanding. The second one begins from January 2008 to December 2012 including the 'subprime' crisis and the European debt crisis. The last period covers a period of economic recovery is from 2013 to 2015.

Moreover, the ad hoc hypothesis of parameters stability implies that the Betas are similar overtime. To confirm or reject this hypothesis, we use forwards rolling regressions with a window of 260 days (1 year in business trading days). As example, figure 2 illustrates the betas estimated by rolling year for AXA. By comparing the classic constant beta of table 1, we can visualize the instability phenomenon of the beta, and this situation persists for all stocks.

We can make equities classification according to characteristics of their

TABLE 1 GLS estimates

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Accor	0.99	24.85	0.000403	1.73	0.47	13.56	16.49	5812.89
Airbus	0.95	30.9	0.000363	1.11	0.35	13.81	0.83	106016
Alcatel	1.22	36.93	−0.000446	−0.95	0.32	1.34	34.01	14326.7
Air Liquide	0.83	29.81	0.000409	2.9	0.65	13.11	57.43	6729.28
AXA	1.51	31.74	0.000425	1.81	0.68	21.07	62.13	41993.2
BNP	1.39	13.38	0.00012	0.46	0.61	29.64	311.01	36979.6
Bouygues	1.06	34.65	0.000157	0.63	0.51	0.3	57.04	1774.4
CA	1.44	21.91	−0.000141	−0.47	0.56	13.97	161.13	7641.56
Carrefour	0.9	26.56	$-3.39e^{-5}$	−0.14	0.48	2.83	76.2	3926.46
Danone	0.65	31.47	0.000277	1.64	0.41	9.19	231.21	4778.36
Essilor	0.54	13.16	0.000501	2.64	0.31	11.92	105.44	16498.4
GDF	0.94	21.81	$9.33e^{-5}$	0.47	0.5	30.29	20.3	164186
Gemini	1.04	41.79	0.000476	1.86	0.48	10.45	14.25	2722.26
St-Gobain	1.34	29.81	$3.65e^{-5}$	0.18	0.67	12.37	222.05	15224.2
L'Oréal	0.72	22.52	0.000346	2.03	0.48	26.83	40.74	4951.32
LVMH	1	36.81	$3.48e^{-4}$	1.91	0.62	13.24	38.34	10867.6
Michelin	1.08	24.89	0.000281	1.29	0.49	13.96	29.17	3655.4
Orange	0.73	18.83	$3.50e^{-5}$	0.16	0.43	17.7	37.81	4480.43
PSA	1.17	27.4	−0.000211	−0.53	0.39	7.69	116.57	1614.53
Publicis	0.72	43.9	0.000306	1.52	0.43	28.88	70.5	1972.62
Renault	1.36	30.46	0.00019	0.55	0.55	12.36	122	2371.17
Ricard	0.69	14.87	0.000327	1.77	0.36	31.55	143.78	8181.13
Schneider	1.24	52.97	0.000412	2.13	0.68	25.79	41.25	976.5
Sodexo	0.64	16.67	0.000502	2.6	0.36	10.21	21.25	9219.78
SG	1.47	16.48	−0.000106	−0.39	0.56	65.51	304.57	13567.8
Technip	1.05	27.25	0.000113	0.35	0.39	17.01	30.72	6149.54
Total	0.93	20.68	0.000183	1.19	0.67	11.82	155.56	2095.14
Veolia	0.92	24.27	$2.78e^{-5}$	0.09	0.39	8.31	1.58	160406
Vinci	1.11	27.05	0.000434	2.62	0.67	5.19	208.29	5024.85
vivendi	0.79	47.02	$9.86e^{-5}$	0.52	0.52	4.9	16.85	7732.78

NOTES Column headings are as follows: (1) GLS, (2) beta, (3) *t*-statistics, (4) constant, (5) *t*-statistics, (6) R^2, (7) Ljung-Box test, (8) ARCH-LM test, (9) Jarque-Bera test. At 5% level of risk, column (7): $\chi^2(5) = 11.1$, column (8): $\chi^2(2) = 5.99$, colum (9): $\chi^2(2) = 5.99$. At 5% risk level, Fisher statistic is 3.85 and all R^2 are significantly different from zero.

betas volatility. With the Rolling Betas we can improve more precisely the systematic risk measure. With the following considerations we can characterise the volatility's degree of the Beta (table 2).

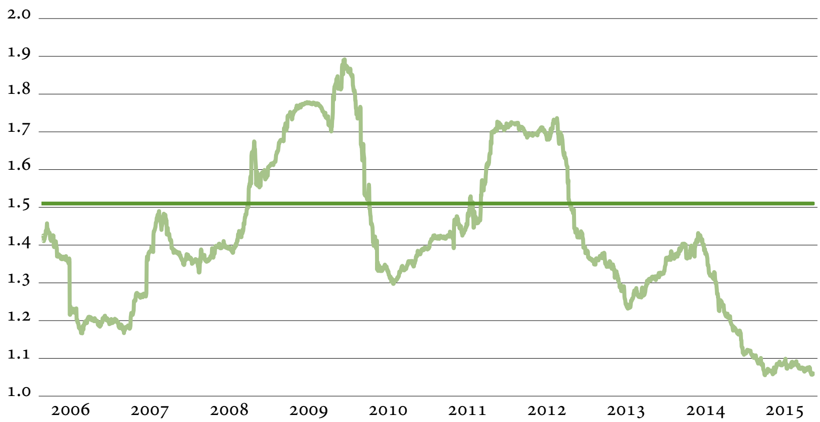

FIGURE 2 Rolling Betas of AXA (light – rolling Betas, dark – classic Beta)

- The estimated betas are tested (equality to 1) and classified by their standard deviation. We create three groups classified by their degree of volatility (the first three columns of the table).
- To select the betas equal to 1, the t-statistics is sometimes insufficient because the betas fluctuate around 1 without stabilizing. We count in the following 3 columns, the percentages of beta higher, lower or equal than 1.
- In the last three columns, we put down the percentages of beta inside or outside the confidence interval of the beta estimated by OLS for characterizing the volatility of betas.

First, we note that the number (and the nature) of equities in each category is different according to the period. It supports the hypothesis of market's line instability over time. For the first period, we notice a fair repartition of equities in each class and there is consequently a fair diversity of profiles. However in crisis time, there are few equities with betas equal to 1 and more stocks with high betas. Furthermore, Betas are more volatile during the crisis which ensues a pronounced non-robustness and a lower percentage of betas inside the confidence interval. A stock is considered robust if the majority of its rolling betas are in the same category of the classic beta.

It is also possible to create syntheses by economic sector: the financial sector has mostly betas greater than 1 but volatile (such as automotive sector). In the other side, the media-advertising industry has mainly betas lesser than 1 with a low volatility. With these tables it is possible to

TABLE 2 Volatility of Beta for the Overall Period

Groups	Stocks	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
(a)	Publicis	0.72	−17.93	0.07	100.00	0.00	0.00	51.89	23.66	24.45
	GDF	0.94	−3.57	0.09	58.80	40.94	0.27	44.04	25.03	30.93
	Vivendi	0.79	−15.26	0.11	93.18	3.11	3.72	35.19	16.72	48.10
	Air Liquide	0.83	−14.93	0.11	83.40	16.60	0.00	34.65	8.78	56.57
	Ricard	0.69	−17.89	0.11	100.00	0.00	0.00	76.73	7.75	15.52
	L'Oréal	0.72	−19.97	0.12	97.82	2.19	0.00	41.24	12.77	45.99
	Danone	0.65	−23.84	0.12	99.62	0.38	0.00	30.85	49.06	46.99
	Total	0.93	−5.77	0.13	60.98	24.49	14.53	49.41	12.50	38.10
	Sodexo	0.64	−22.65	0.16	97.39	2.11	0.50	60.60	8.36	31.04
	Airbus	0.95	−2.00	0.17	47.79	20.55	31.66	40.20	19.02	40.78
	Veolia	0.92	−3.90	0.20	80.38	2.77	16.86	68.69	11.89	19.43
	Essilor	0.54	−31.03	0.21	93.48	6.52	0.00	40.05	6.21	53.74
	Carrefour	0.90	−5.75	0.21	56.42	13.91	29.67	53.35	2.61	44.04
	Orange	0.73	−17.60	0.21	70.83	12.46	16.71	36.22	11.81	51.97
(b)	LVMH	1.00	−0.21	0.11	27.75	51.78	20.47	36.41	29.32	34.27
	Gemini	1.04	1.80	0.19	36.60	18.86	44.54	45.61	10.62	43.77
	Accor	0.99	−0.52	0.21	59.41	8.93	31.66	40.20	19.02	40.78
(c)	Schneider	1.24	14.72	0.14	4.40	8.06	87.54	41.81	20.05	38.14
	Bouygues	1.06	3.02	0.14	32.00	20.09	47.91	43.30	20.98	35.72
	PSA	1.17	6.04	0.15	12.46	12.11	75.43	39.06	26.44	34.50
	Vinci	1.11	7.85	0.16	19.66	19.86	60.48	50.94	18.21	30.85
	Renault	1.36	15.48	0.17	0.00	4.22	95.78	55.62	17.56	26.83
	Alcatel	1.22	6.66	0.17	8.43	17.79	73.78	41.17	23.80	35.03
	Michelin	1.08	3.70	0.17	24.65	30.29	45.07	51.28	18.33	30.39
	St-Gobain	1.34	19.45	0.18	3.68	11.27	85.05	42.81	39.56	17.63
	AXA	1.51	26.20	0.21	0.00	0.00	100.00	69.49	4.14	26.37
	BNP	1.39	18.84	0.21	0.00	0.00	100.00	53.00	15.65	31.35
	Technip	1.05	1.99	0.22	35.42	21.51	43.08	44.57	21.55	33.88
	CA	1.44	18.46	0.23	0.81	2.04	97.16	49.37	29.43	29.44
	SG	1.47	19.28	0.30	0.00	0.00	100.00	52.08	6.60	41.32

NOTES Colum/row headings are as follows: (1) classic beta, (2) *t*-statistics to 1, (3) standard deviation of beta, (4) percentage of beta < 1, (5) percentage of beta = 1, (6) percentage of beta > 1, (7) percentage of lower CI, (8) percentage in CI, (9) percentage of upper CI, (a) betas always less than 1, (b) betas always greater than 1, (c) volatile betas relative to 1.

construct a stock overview useful for a portfolio manager. We improve its investment opportunities by adding a frequency approach.

Table 3 illustrates our conclusions about the betas volatility for 3 equi-

TABLE 3 Synthesis of the Betas Volatility

Degree of Volatility	Overall Period	Period 1	Period 2	Period 3
LVMH	Medium	High	Low	High
AXA	High	Medium	High	High
Essilor	High	Low	Medium	Medium

ties (we built tables for all stocks): AXA with a high beta, LVMH with a beta equal to 1 and Essilor with a low beta.

A Time-Frequency CAPM: A Wavelets Approach

Wavelets are an extension of the spectral analysis of Fourier. We can evaluate the temporal evolution of the spectral values for different frequencies. A mother wavelet Ψ, in L^2 space, with null and standardized mean is defined by the following equation (we use Mallat's (2001) notation):

$$\int_{-\infty}^{+\infty} \Psi(t)dt = 0. \tag{3}$$

We define the wavelet family which regrouping all translated (by a parameter τ) and dilated (by a scale parameter s) versions of the mother wavelet:

$$\Psi_{\tau,s}(t)\frac{1}{\sqrt{s}}\Psi\left(\frac{t-\tau}{s}\right). \tag{4}$$

The continuous wavelet transform of a temporal function $x(t)$ (realization of a random function of the same name) by a wavelet $\Psi(t)$ of scale s and position τ, gives the following convolution:

$$W_x(\tau, s) = \frac{1}{\sqrt{s}}\int_{-\infty}^{+\infty} x(t)\Psi^{*}\left(\frac{t-\tau}{s}\right)dt, \tag{5}$$

Ψ^{*} is complex conjugate of $\Psi(t)$.

We can rebuild the original chronic with the wavelets coefficients $W_x(\tau, s)$. This is the inverse wavelet transform:

$$x(t) = \frac{1}{C_\Psi}\int_{-\infty}^{+\infty} \Psi_{\tau,s}(t)W_x(\tau, d)\frac{d\tau ds}{s^2}. \tag{6}$$

With $C_\Psi = \int_0^{+\infty} \frac{|\widehat{\Psi(\omega)}|^2}{\omega}$ the condition of existence of the wavelet, ω is the angular frequency and $\widehat{\Psi(\omega)}$ the Fourrier's transform of the imaginary part of $W_x(\tau, s)$.

In this paper, we use a discrete version of the wavelets transform called the Maximal Overlap Discrete Wavelets Transform (MODWT). The dis-

TABLE 4 Frequency Bands and Their Corresponding Days and Months

Frequency bands	Frequency days		Frequency months	
	Lower limits	Upper limits	Lower limits	Upper limits
D1	2	4	0.09	0.18
D2	4	8	0.18	0.36
D3	8	16	0.36	0.73
D4	16	32	0.73	1.45
D5	32	64	1.45	2.91
D6	64	128	2.91	5.82
D7	128	256	5.82	11.64
D8	256	512	11.64	23.27
D9	512	1024	23.27	46.55
D10	1024	2048	46.55	93.09
D11	2048	4096	93.09	186.18
S11	4096	–	186.18	–

cretization is useful to reduce the calculation time. It provides additive wavelet decomposition described by the following equation:

$$X_t = A_{J,t} + \sum_{i=1}^{i=J} D_{i,t}. \tag{7}$$

where x_t is our starting signal (time series) constituted by an approximation of the trend noted $A_{J,t}$ and a sum of subseries $D_{i,t}$ (for $i = 1 \dots J$). These subseries are additional details added to the basic approximation; they are assimilating to the accuracy of the decomposition and the index J corresponds to the optimal number of details. It is calculated as

$$J = \frac{Ln(n)}{Ln(2)},$$

where n is the number of observations.

We are able to study the behaviour of the signal by frequency levels linked with time localization. Table 4 regroups the number of frequency bands and the corresponding days and months. We note that more the time scale increases more the band's variance is lower than high frequency bands.

FREQUENCY BETA ESTIMATE

The estimation of the frequency market's line beta parameter is performed on the bands D1 to D6 corresponding to trading behaviours from 2 days to 6 months. All betas are significant but we note the use of wavelet does not resolve the heteroscedasticity and autocorrelation of the residuals. In contrast, the non-normality is reduced at low frequencies for few actions. Their R^2s are different according to the stock. In general, they are high (around 0.6–0.7) for strong betas stocks. We can't present all the results because of the large number (table 5 illustrates the frequency betas for the 30 equities).

According to the investment horizon the Betas are different. We distinguish equities with a classic beta less than 1 (respectively greater than 1) and all their frequency betas are in the same category. We show that there are stocks with volatile frequency betas compared to 1 and regrouped in the third category. There are also more stocks with all frequency betas less than 1 and their standard deviations are lower than high betas equities.

In order to appreciate the effect of the economic environment on Systematic Risk, we reiterate the wavelet methodology on the 3 previous subperiods. We add that differences between the frequency betas and the classic beta are more significant during the 'crisis' period. The frequency betas could appreciate more accurately the systematic risk during this period. Moreover, the frequency long-run betas are totally different of the classic beta, proving the importance of using wavelets to measure systematic risk for a long-term investment.

We compare the Frequency Betas with the Classic Beta by a Principal Component Analysis. By placing the classic beta as supplementary variable, its projection on the factorial axes is merged with high frequencies betas especially with D1 and D2 (figure 3). So, when managers use the market's line to manage their portfolios (the classic beta), they implicitly have a short-term trading behaviour. The PCA clearly reflects the diversity of choices that can be made according to the investment horizon. The projection of stocks on this factorial component clarifies the classification realized in table 5. Axis 1 (91.27% of the total variance) opposes stocks with a beta greater than 1 for all wavelets scales (as Société Générale) to low beta stocks (beta less than 1) in most frequency bands (as Essilor). Axis 2 (5.57% of the total variance) distinguishes stocks according to their frequency dynamics.

Another method to compare the frequency betas with classic betas is a significance test on the difference between these two Betas. Table 6 summarizes percentages of significant different betas for the three subperiods and the total period.

For D1 and D2 bands, differences are not significant for the majority of

TABLE 5 Frequency Betas for the Global Period

Groups	Stocks	D1	D2	D3	D4	D5	D6	(1)	(2)
(a)	Carrefour	0.9	0.92	0.85	0.95	0.88	0.75	0.07	0.9 (<1)
	Danone	0.68	0.67	0.57	0.46	0.6	0.73	0.10	0.65 (<1)
	Essilor	0.55	0.5	0.45	0.6	0.72	0.48	0.10	0.54 (<1)
	GDF	0.93	0.98	0.87	0.95	0.96	0.8	0.07	0.94 (<1)
	L'Oréal	0.76	0.73	0.61	0.55	0.7	0.81	0.10	0.72 (<1)
	Orange	0.75	0.73	0.68	0.64	0.63	0.59	0.06	0.72 (<1)
	Publicis	0.69	0.73	0.76	0.83	0.82	0.86	0.07	0.72 (<1)
	Ricard	0.66	0.71	0.73	0.69	0.8	0.99	0.12	0.69 (<1)
	Sodexo	0.61	0.69	0.66	0.65	0.65	0.61	0.03	0.64 (<1)
	Vivendi	0.8	0.79	0.76	0.73	0.72	0.75	0.03	0.79 (<1)
(b)	Alcatel	1.11	1.24	1.41	1.34	1.66	1.75	0.25	1.22 (>1)
	Axa	1.45	1.56	1.54	1.61	1.52	1.77	0.11	1.51 (>1)
	BNP	1.37	1.37	1.49	1.6	1.05	1.07	0.22	1.39 (<1)
	CA	1.37	1.47	1.59	1.7	1.29	1.52	0.15	1.44 (>1)
	Renault	1.22	1.39	1.56	1.62	1.47	2.01	0.27	1.35 (>1)
	Schneider	1.25	1.24	1.18	1.24	1.15	1.08	0.07	1.24 (>1)
	SG	1.38	1.45	1.7	1.71	1.54	1.91	0.20	1.47 (>1)
	St-Gobain	1.31	1.35	1.35	1.5	1.44	1.49	0.08	1.34 (>1)
(c)	Airbus	0.93	0.94	1.05	0.97	1.16	1.19	0.11	0.95 (<1)
	Gemini	0.98	1.08	1.1	1.07	1.23	1.18	0.09	1.04 (=1)
	LVMH	0.98	1.02	1.04	1	1.05	0.97	0.03	1 (=1)
	Accor	0.91	1.06	1.11	1.07	1.05	1.18	0.09	0.99 (=1)
	Michelin	1.01	1.11	1.23	1.12	1.06	1.16	0.08	1.08 (>1)
	PSA	1.04	1.24	1.4	1.36	1.39	1.08	0.16	1.17 (>1)
	Technip	0.99	1.06	1.2	1.16	1	1.05	0.09	1.05 (>1)
	Total	0.94	0.94	0.87	0.95	0.99	0.83	0.06	0.93 (<1)
	Veolia	0.86	0.86	1.06	1.11	1.17	1.34	0.19	0.92 (<1)
	Vinci	1.12	1.14	1.12	1.04	0.91	1.04	0.09	1.11 (>1)
	Bouygues	1.06	1.04	1.08	1.11	1	1.11	0.04	1.06(>1)
	Air liquide	0.87	0.81	0.78	0.78	0.83	0.6	0.09	0.83 (<1)

NOTES Column/row headings are as follows: (1) betas SD (standard deviation of frequency betas of the corresponding stock), (2) classic beta, (a) betas always less than 1, (b) betas always greater than 1, (c) volatile betas relative to 1. Because of large numbers of Betas we don't present the results for the 3 sub-periods.

stocks at short-term horizons. The low frequency bands have 60 to 80% of the frequency beta significantly different of classic betas. This result confirms our previous hypothesis, the use of classic betas supposes the

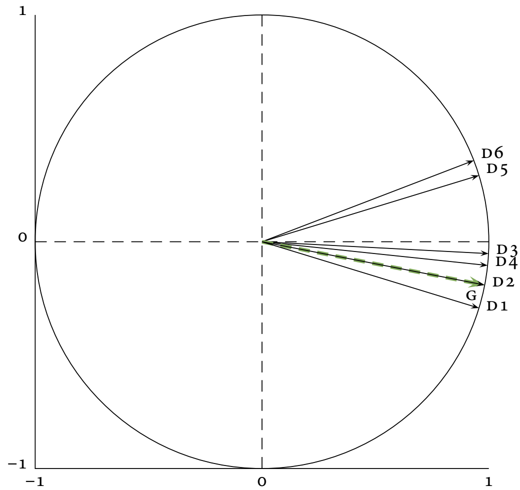

FIGURE 3

Correlation Circle with the Classic Betas in Supplementary Variable (G)

TABLE 6 Percentages of Significantly Different Betas

Period	D1	D2	D3	D4	D5	D6
Period 1	0.00	3.33	16.67	33.33	60.00	73.33
Period 2	16.67	10.00	60.00	70.00	53.33	80.00
Period 3	6.67	6.67	40.00	46.67	56.67	63.33
Overall Period	30.00	10.00	66.67	60.00	63.33	80.00

systematic risk over short horizons. It is strongly dependent of the chosen frequency (the investment horizons). Stocks are differentially sensitive to the market and investors should include the temporal volatility in their procedure but also the frequency volatility of systematic risk.

ANALYSIS OF THE FREQUENCY BETA INSTABILITY

We study the volatility of frequency betas using rolling regressions with a 260-days window. Given the large number of stocks and frequency bands, we retain only D1 bands (for short horizons) and D6 (for long investment) of the previous selected equities. Table 7 records results for the frequency betas using the same methodology presented in the section on classic beta estimate.

These results extended to all stock can be synthesized by an investor based on their own preferences and criteria which they consider more relevant. For example, high volatility is considered by a standard deviation of rolling betas greater than 0.15 and low volatility by standard devi-

TABLE 7 Frequency Betas Volatility

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
LVMH D1	0.98	−1.62	0.1	37.03	41.78	21.2	0.00	36.41	29.32	34.27
Axa D1	1.45	24.44	0.18	0.00	0.00	100.00	0.00	65.93	7.36	26.72
Essilor D1	0.55	−29.72	0.23	88.54	10.58	0.88	0.00	40.02	3.91	56.08
LVMH D6	0.97	−2.60	0.38	45.34	7.67	46.99	0.00	36.41	29.32	34.27
Axa D6	1.77	36.99	0.59	16.86	3.87	79.26	0.00	70.37	2.61	27.02
Essilor D6	0.48	−34.06	0.43	88.62	3.60	7.78	15.44	60.83	4.60	34.57
LVMH	1	−0.21	0.11	27.75	51.78	20.47	0.00	36.41	29.32	34.27
AXA	1.51	26.20	0.21	0.00	0.00	100	0.00	69.49	4.14	26.37
Essilor	0.54	−31.03	0.21	93.48	6.52	0.00	0.00	40.05	6.21	53.74

NOTES Colum headings are as follows: (1) overall period, (2) frequency betas, (3) *t*-statistics 1, (4) standard deviation of beta, (5) percentage of beta < 1, (6) percentage of beta = 1, (7) percentage of beta > 1, (8) percentage of beta ≤ 1, (9) percentage of lower CI, (10) percentage in CI, (11) percentage of upper CI.

TABLE 8 Syntheses of Frequency Betas Volatility

Period		Classic Beta	Beta D1	Beta D6
Overall Period	LVMH	Medium	Medium	High
	AXA	High	High	High
	Essilor	High	High	High
Period 1	LVMH	High	High	High
	AXA	Medium	Low	High
	Essilor	Low	High	High
Period 2	LVMH	Low	Low	High
	AXA	High	Medium	High
	Essilor	Medium	Medium	High
Period 3	LVMH	High	Medium	Medium
	AXA	High	Medium	High
	Essilor	Medium	Medium	High

ation less than 0.1. By analysing the characteristics of betas values, we can create a summary described in table 8.

The beta volatility is more important for the total period than the three sub-periods. Betas for medium frequency D6 are more volatile than the short-term betas (D1). We can make an equities overview based on the time-frequency characteristics useful for appreciate the Systematic Risk related to our investment horizon.

Conclusion and Discussion

The market's line estimation on the period 2005–2015 presents statistical anomalies cause of the autocorrelation and the heteroscedasticity in residuals. The volatility of beta, illustrated by rolling regressions, is probably the most important reason of the residual nature.

Due to the characteristics of frequency betas volatility, we realize an equity's classification useful to appreciate the systematic risk adding the heterogeneity hypothesis of agents. It is studied by time-frequency estimation of the market's line over the period and sub-periods. There is a differentiation of betas according to the frequency and the selected period. If the manager uses the classic beta to measure his systematic risk, he makes 'mistakes' omitting the impact of the investment horizon on the risk.

To illustrate this conclusion, Veolia stock on the period 2005–2015 has a beta equal to 0.92. This stock attenuates the market fluctuations by losses (or earnings) which are lower than the market. The risks profile is more defensive-tracker useful in crisis. It is established without taking into consideration the different investment horizons. To estimate the frequency betas by using the wavelets, we note that the short-run beta is 0.86 (investment horizon of 2 to 4 days so it is the D1 bands) while the long-run beta (for an investment horizon of 3 to 6 months) is 1.34. If the portfolio manager uses the classic beta of this stock for investing in short-term, he minimizes his defensive performance. However, if he invests in long-run he has a beta greater than one and therefore the stock became more aggressive.

The distinction by frequency betas multiplies the investment choices of agents. It provides a risk measure more in line with their appetencies and behaviour. As we can find, the long-run betas are significantly different from the classic beta and this is true for the vast majority of equities. Following this example, Veolia has short-term betas slightly below the classic betas. In contrast, the mid and long run betas are significantly higher than the classic betas. This result confirms the previous observation: short-term betas are overestimated but long-term betas are underestimated (the reverse is possible for example with Vivendi). We conclude that the choice of agents is biased and this fact causes additional risks.

The advantage of using wavelet decomposition to limit these 'mistakes' on betas is significant. The rolling regressions specifically characterize this instability by analysing the volatility leading to an overview describing the dynamics of rolling betas. This classification is repeated for each period. By analysing and comparing our results, we show that the systematic risk and its volatility are significantly different according to the investment horizon and the period. It provides additional degrees of freedom to make investment choices.

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